Differentiation in n-d

In 1-d we have $f : U \to \mathbb{R}$ with $U \subseteq \mathbb{R}$ open. Then in n-d a function is a map $f : U \to \mathbb{R}$ with $U \subseteq \mathbb{R}^n$ open and a map is a function $f : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^n$ open.

1. Remark: A map $U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^n$ open is uniquely determined by $f_1, f_2, \ldots, f_n : U \to \mathbb{R}$ by looking at coordinates of $\mathbb{R}^n$.

1.1. Examples:

1. $f : \mathbb{R} \to \mathbb{R}^2$ with $t \mapsto (\cos t, \sin t)$
   In this example, $f_1(t) = \cos t$, $f_2(t) = \sin t$.
2. $f : \mathbb{R}^2 \to \mathbb{R}$
   Suppose $f(x, y) = e^{\sin x \cdot \cos^2 x} + x + y$. We cannot write an x function and a y function since we only have one output.

2. Generalising 1-d

We have a limiting slope interpretation. But consider the case of $f : \mathbb{R}^n \to \mathbb{R}$, here we have:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{\text{a number}}{n\text{-tuple}} = \text{undefined}$$

To remedy this we may instead take the limit (wlog consider 2-d) $\lim_{(x,y) \to (x_0, y_0)}$ but this doesn't work (consider $f(x, y) = 2x + y$). Thus we generalise not the limiting slope interpretation but instead the linear approximation interpretation.

2.2. Examples:

1. $f(x) = (x + 1)^3$ at $x = 0$
   We expand $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$. Here the first two terms are our error term and the last two are our linear approximation $L(x)$. Notice then that $\lim_{x \to 0} \frac{f(x) - L(x)}{x} = 0$.
2. $f(x) = e^x$ at $x = 0$
   We expand $e^x = 1 + x + \frac{x^2}{2} + \cdots$, here $L(x) = 1 + x$.

3. Remark: In general the linear approximation is the function of the form $C + a_1x_1 + a_2x_2 + \cdots + a_nx_n$**.**

4. Definition: Let $f : U \to \mathbb{R}^m$ **with** $U \subseteq \mathbb{R}^n$ **open. We say** f **is differentiable at** $x_0 \in U$ **if there exists a linear map** $T : \mathbb{R}^n \to \mathbb{R}^m$ **s.t.**

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - T(x - x_0)||}{||x - x_0||} = 0.$$

If the limit exists, we say the differential matrix $Df(x_0) = T$ **is a linear map.**

5. Remark: Recall that if $m = 1$ **we say** $T = [a_1, \ldots, a_n]$ **is a linear map** $\mathbb{R}^n \to \mathbb{R}$**.**